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LETTER TO THE EDITOR

Exact results on Landau-level broadening

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Abstract. For an electron in the plane subjected to a perpendicular constant magnetic field and a homogeneous random potential we investigate the averaged density of states restricted to an arbitrary Landau level. We calculate the exact width for an arbitrary random potential. For general Gaussian random potentials we prove the existence of Gaussian tails and determine the decay constants. Furthermore, we derive Jensen-type lower and upper bounds to expectation values of convex functions with respect to the restricted density of states. Using similar bounds we estimate the effect of Landau-level mixing.

The realization [1] of nearly ideal two-dimensional electron systems [2] has stimulated a tremendous amount of experimental and theoretical research during the past two decades [1, 3]. One basic tool in studying the properties of these systems is the application of a magnetic field [4]. For the explanation of recent experiments [5–7] the early theories [8] for the gas of non-interacting electrons in two dimensions under the influence of a perpendicular constant magnetic field have been found to be insufficient [9]. This gas is characterized by the one-electron Hamiltonian given, in the Schrödinger representation, as

$$H_0 := \frac{1}{2m}\left(\frac{\hbar\partial}{i\partial x_1} - \frac{eB}{2}x_2\right)^2 + \frac{1}{2m}\left(\frac{\hbar\partial}{i\partial x_2} + \frac{eB}{2}x_1\right)^2. \tag{1}$$

Here $x := (x_1, x_2)$ are Cartesian coordinates of the infinite plane, $\hbar$ is Planck's constant, e is the elementary charge, m is the (effective) mass of the (spinless) electron, and B the strength of the magnetic field.

A more adequate description of the observed phenomena should become possible, when extending the model (1) by the addition of a static and homogeneous random potential V mimicking the interaction with quenched disorder. In particular, the resulting Hamiltonian $H := H_0 + V$ is believed [9] to provide a minimal model for the explanation of the integral quantum Hall effect [5].

As well as for the understanding of this model as for the rating of its capability for explaining experiments, the averaged density of states (per area) $D(\varepsilon) := \langle x|\overline{\delta(\varepsilon - H)}|x\rangle$ is a quantity of basic importance. Here the overbar denotes the average with respect to the probability distribution of V.

While over the years many approximations to D, for example [10–14], have been devised, only a few exact results are available. The density of states D is exactly known for a random potential which is either constantly correlated [15, 16] or Cauchy–Lorentz white noise [16, 17]. Moreover, for Gaussian random potentials the leading low-energy behaviour of D is known to be Gaussian [18], if $\overline{V(x)^2} < \infty$.

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Often the Hamiltonian H is restricted to the eigenspace of the unperturbed Hamiltonian H_0 belonging to the nth eigenvalue $\varepsilon_n := (2n+1)\hbar eB/2m$, that is, the nth Landau level ($n = 0, 1, 2, \ldots$). The corresponding projection operator E_n is given in the position representation as

$$\langle x|E_n|x'\rangle = (2\pi l^2)^{-1} \exp\{[2i(x_1x_2' - x_2x_1') - (x-x')^2]/4l^2\} L_n((x-x')^2/2l^2) \tag{2}$$

where L_n is the nth Laguerre polynomial [19] and $l := (\hbar/eB)^{1/2}$ is the magnetic length.

The simplification implied by this neglect of Landau-level mixing, which is believed to be justifiable for high magnetic fields, see [18] and below, allows for further exact results, namely on the averaged restricted density of states (per area)

$$D_n(\varepsilon) := \langle x|\overline{E_n\delta(\varepsilon - E_nHE_n)E_n}|x\rangle. \tag{3}$$

This density equals the spectral density of a quantum system with one degree of freedom characterized by an appropriate random Hamiltonian [20]. The most remarkable result [17, 21] in this context is that D_0 is known for a general white-noise potential.

In the present letter, for the first time, exact results on D_n are presented for general n and a random potential with arbitrary correlation lengths.

For notational convenience we will assume, without loss of generality, that the homogeneous random potential has zero mean $\overline{V(0)} = 0$, and call its covariance $\overline{V(x)V(0)} =: C(x)$. Under these circumstances $2\pi l^2 D_n$ is a probability density on the real line with first moment ε_n and variance

$$\sigma_n^2 := 2\pi l^2 \int \mathrm{d}\varepsilon\, D_n(\varepsilon)(\varepsilon - \varepsilon_n)^2 = 2\pi l^2 \int \mathrm{d}^2x |\langle 0|E_n|x\rangle|^2 C(x). \tag{4}$$

For the derivation of (4) one observes $\sigma_n^2 = 2\pi l^2 \langle 0|\overline{E_nVE_nVE_n}|0\rangle$ and uses the formula

$$\langle 0|E_n|x+x'\rangle\langle x+x'|E_n|x'\rangle = \langle x|E_n|0\rangle\langle -x'|E_n|x\rangle. \tag{5}$$

The simple-looking relation (4) is one of our major results, because σ_n is naturally interpreted as the width of the nth Landau level broadened by an arbitrary random potential.

Remarks

The quantity σ_n has earlier appeared in the literature [11, 13, 18], but has not been identified with the exact width of D_n there.

The variance σ_n^2 never exceeds the single-site variance of the random potential. More generally $\sigma_n^2 \leq \min\{C(0), \sup_k \tilde{C}(k)/2\pi l^2\}$. Here $\tilde{C}(k) := \int \mathrm{d}^2x\, C(x) \exp\{-\mathrm{i}kx\}$ is the Fourier transform of C which is non-negative.

If both C and $\tilde{C}$ are bounded, one can show that the broadening vanishes in the high Landau-level limit, that is, $\lim_{n\to\infty} \sigma_n^2 = 0$. This, in turn, implies by the Chebyshev inequality that the density of states equals the unperturbed one asymptotically $\lim_{n\to\infty} D_n(\varepsilon + \varepsilon_n) = \delta(\varepsilon)/2\pi l^2$. This result demonstrates that the non-trivial densities calculated in the high Landau-level limit for some white-noise potentials [22] are the exception rather than the rule.

For the often discussed example of the Gaussian covariance

$$C(x) = C(0)\,\mathrm{e}^{-x^2/2\lambda^2} \tag{6}$$

the level width can explicitly be calculated as a function of the correlation length λ to

$$\sigma_n^2(\lambda^2) = C(0)\,\frac{\lambda^2}{\lambda^2 + l^2}\left(\frac{\lambda^2 - l^2}{\lambda^2 + l^2}\right)^n P_n\left(\frac{\lambda^4 + l^4}{\lambda^4 - l^4}\right). \tag{7}$$

Here P_n denotes the nth Legendre polynomial [19]. In this case the above-mentioned upper bound on σ_n^2 can be sharpened to $\sigma_n^2(\lambda^2) \leq \sigma_0^2(\lambda^2)$; there is even convincing numerical evidence [11] for $\sigma_{n+1}^2(\lambda^2) \leq \sigma_n^2(\lambda^2)$. We note that $\sigma_n^2(\lambda^2)$ is increasing in λ^2 and obeys the reciprocity relation $\sigma_n^2(\lambda^2) = (\lambda^2/l^2)\sigma_n^2(l^4/\lambda^2)$.

While we conjecture that the monotonicity properties of the widths $\{\sigma_n\}$, found for the Gaussian covariance, hold true for all covariances which are decreasing as a function of $|x|$, there are also covariances leading to rather strange broadening effects. For example, the oscillating covariance $C(x) = C(0)J_0(\sqrt{2}|x|/\lambda)$, with J_0 denoting the zeroth Bessel function of the first kind [19] and a given correlation length $\lambda > 0$, yields $\sigma_n^2 = C(0)\exp\{-l^2/\lambda^2\}[L_n(l^2/\lambda^2)]^2$. Choosing l^2/λ^2 as a zero of $L_n (n \geq 1)$ the nth Landau level is not broadened at all. Moreover, for given $n \geq 1$ one can find l^2/λ^2 such that $\sigma_0^2 < \sigma_n^2$. Finally, although $\tilde{C}(k) = 4\pi\lambda^2 C(0)\delta(\lambda^2 k^2 - 2)$ is not bounded, one has $\sigma_\infty^2 = 0$.

We close the remarks on the exact width σ_n by advancing the results for the two extreme situations of the spatial extent of correlations, separately. For the constant covariance $C(x) = C(0)$ one gets $\sigma_n^2 = C(0)$, whereas for the delta-covariance $C(x) = \nu^2\delta(x)$ one finds $\sigma_n^2 = \nu^2/2\pi l^2$.

After the discussion of the level width σ_n as the root-mean-square of the probability density $2\pi l^2 D_n$ we now turn to the expectation value of an arbitrary convex function $\varepsilon \mapsto f(\varepsilon)$ with respect to $2\pi l^2 D_n$. For general f we are only able to give lower and upper bounds

$$\int d\varepsilon\, \rho_{n,\psi}(\varepsilon) f(\varepsilon) \leq \int d\varepsilon\, D_n(\varepsilon) f(\varepsilon) \leq \int d\varepsilon\, \rho_n^{qc}(\varepsilon) f(\varepsilon). \tag{8}$$

Here

$$\rho_{n,\psi}(\varepsilon) := (2\pi l^2)^{-1}\overline{\delta(\varepsilon - \varepsilon_n - \langle\psi|E_n V E_n|\psi\rangle)} \tag{9}$$

for each Hilbert-space vector $|\psi\rangle$ with $\langle\psi|E_n|\psi\rangle = 1$, and

$$\rho_n^{qc}(\varepsilon) := (2\pi l^2)^{-1}\overline{\delta(\varepsilon - \varepsilon_n - V(0))} \tag{10}$$

may serve as simple approximations to D_n (see, e.g. [14]).

The lower bound in (8) follows from the identity [17, 20]

$$\overline{E_n f(E_n H E_n) E_n} = E_n 2\pi l^2 \int d\varepsilon\, D_n(\varepsilon) f(\varepsilon) \tag{11}$$

in combination with the trace version [23] of Jensen's inequality

$$\mathrm{Tr}(Ef(EAE)E) \leq \mathrm{Tr}(Ef(A)E) \qquad A = A^\dagger \qquad E = E^\dagger E \tag{12}$$

for the choice $A = E_n H E_n$ and $E = E_n|\psi\rangle\langle\psi|E_n$.

For a proof of the upper bound in (8) we can assume $f(\varepsilon_n) = 0$ without loss of generality. We apply (12) to $A = \varepsilon_n + V_r$ and $E = E_n$, where $V_r(x) := V(x)\Theta(r - |x|)$ is the random potential cut-off to a disk of radius $r > 0$ centred at the origin. Finally, after averaging we divide by πr^2 and pass to the limit $r \to \infty$.

Remarks

By definition, ρ_n^{qc} is a quasiclassical approximation and, therefore, the appropriate adaptation of the early proposal in [24].

For given f the lower bound in (8) may be optimized with respect to $|\psi\rangle$. As a consequence, the approximation $\rho_{n,\psi}$ may be seen as the appropriate adaptation of the proposals in [25].

For the special choice $f(\varepsilon)=\exp\{-\beta\varepsilon\}(\beta\geq 0)$ and $|\psi\rangle=E_n|0\rangle(2\pi l^2)^{1/2}$ the inequalities (8) have already appeared in [18].

For the choice $f(\varepsilon)=2\pi l^2(\varepsilon-\varepsilon_n)^2$ the first inequality in (8) implies that the variance

$$\gamma_{n,\psi}^2 := 2\pi l^2 \int \mathrm{d}\varepsilon\, \rho_{n,\psi}(\varepsilon)(\varepsilon-\varepsilon_n)^2 = \int \mathrm{d}^2x \int \mathrm{d}^2x' |\langle\psi|E_n|x\rangle|^2 C(x-x') |\langle x'|E_n|\psi\rangle|^2 \tag{13}$$

of $2\pi l^2\rho_{n,\psi}$ underestimates σ_n^2. Additionally, it is possible to show that $\gamma_{n,\psi}^2$ vanishes for all $|\psi\rangle$ in the high Landau-level limit $n\to\infty$, if $\tilde{C}$ is bounded.

Choosing $f(\varepsilon)=\beta^{-1}\ln(1+\exp\{\beta(\mu-\varepsilon)\})$ in (8) leads to bounds on the averaged grand-canonical potential (per area) of non-interacting electrons, with the restricted one-electron Hamiltonian E_nHE_n, at temperature $1/\beta k_B$ and with chemical potential μ. In particular, weakening the lower bound in (8) to $f(\varepsilon_n)/2\pi l^2$ according to Jensen's unequality shows that the inclusion of disorder always lowers the grand-canonical potential.

For the simple random potentials as all the constantly correlated random potentials and the Cauchy–Lorentz white-noise [16] one has $\rho_{n,\psi}=D_n=\rho_n^{qc}$.

From now on we will only consider Gaussian random potentials. Clearly, for these potentials the density $2\pi l^2\rho_{n,\psi}$ is a Gaussian with variance $\gamma_{n,\psi}^2$ centred at the Landau level ε_n. In this case the lower bound in (8) is an increasing function of this variance for all f. Therefore, the best lower bound is universally achieved by the largest variance

$$\Gamma_n^2 := \sup_{\psi} \gamma_{n,\psi}^2. \tag{14}$$

Remarks

For general covariances C the largest variance is restricted by $\sigma_n^4/C(0)\leq\Gamma_n^2\leq\sigma_n^2$.

For the Gaussian covariance (6) the largest variance can explicitly be determined as a function of the correlation length in terms of the width according to

$$\Gamma_n^2(\lambda^2)=[\lambda^2/(\lambda^2+l^2)]\sigma_n^2(\lambda^2+l^2). \tag{15}$$

This results from the fact that the supremum in (14) is realized for the Hilbert-space vector $|\psi_n\rangle$ given in the position representation as $\langle x|\psi_n\rangle\propto(x_1+\mathrm{i}x_2)^n\exp\{-x^2/4l^2\}$. In the subspace of the nth Landau level $|\psi_n\rangle$ is most localized in the sense of minimal position variance. Therefore it seems plausible that $|\psi_n\rangle$ maximizes $\gamma_{n,\psi}^2$. The actual proof, however, requires more work. Since C of (6) is isotropic, it can be based on the inequality $\gamma_{n,\psi}^2\leq\sup_j c_{n,j}$, where

$$c_{n,j} := \frac{1}{2\pi l^2}\int \frac{\mathrm{d}^2\xi}{2\pi}\tilde{C}(\xi/l)\,\mathrm{e}^{-\xi^2}[L_n(\xi^2/2)]^2 L_j(\xi^2)\leq\sigma_n^2 \tag{16}$$

may be interpreted [26] as the jth eigenvalue ($j=0,1,2,\ldots$) of the isotropic two-particle interaction potential $C(x-x')$ with the Hilbert space of each particle restricted to the subspace of the nth Landau level. Since $\gamma_{n,\psi_n}^2=c_{n,0}$, it is sufficient to show $c_{n,j}\leq c_{n,0}$ in order to complete the proof. For the validity of $c_{n,j}\leq c_{n,0}$ for general λ and n we have found strong numerical evidence. So far we only have analytical proofs in the cases (i) $n=0,1$ for general λ and (ii) $C(x)=\nu^2\delta(x)$ for general n. We close the remark on the Gaussian covariance with an analytical curiosity. The mapping

$\zeta \mapsto \Gamma_n^2(\zeta l^2)/\sigma_n^2(\zeta l^2)$ has a stable fixed point at the golden mean $(\sqrt{5}-1)/2$ independent of n.

We conjecture that $\Gamma_n^2=\gamma_{n,\psi_n}^2$ remains true for general isotropic covariances decreasing in $|x|$. For $n=0$ we are able to prove this assertion. The oscillating covariance $C(x)=C(0)J_0(\sqrt{2}|x|/\lambda)$ illustrates that $\Gamma_n^2=\gamma_{n,\psi_n}^2$ is not generally true. For example, this covariance leads to $\gamma_{0,\psi_0}^2<\gamma_{0,\psi}^2$ for $\langle x|\psi\rangle \propto (x_1-\mathrm{i}x_2)\exp\{-x^2/4l^2\}$ provided that $l^2/\lambda^2>2$.

For the constant covariance $C(x)=C(0)$ one immediately gets $\Gamma_n^2=C(0)$, whereas for the delta-covariance $C(x)=\nu^2\delta(x)$ one finds $\Gamma_n^2=(\nu^2/4\pi l^2)(2n)!/(n!2^n)^2$.

For Gaussian random potentials one would intuitively expect that the density of states D_n falls off like a Gaussian for sufficiently large $|\varepsilon-\varepsilon_n|/\sigma_n$. This expectation is supported by the fact that both densities $\rho_{n,\psi}$ and ρ_n^{qc} are Gaussians. In fact, we assert that

$$\lim_{|\varepsilon|\to\infty}\frac{1}{\varepsilon^2}\ln D_n(\varepsilon+\varepsilon_n)=-\frac{1}{2\Gamma_n^2} \tag{17}$$

where the decay constant Γ_n is defined by (14) and (13). This constitutes another major result of the present letter, because it establishes and determines Gaussian tails for arbitrary Gaussian random potentials including Gaussian white noise.

We sketch our proof as follows. Appealing to the appropriate Tauberian theorem to be found, for example, in [27], it is sufficient to show that the two-sided Laplace transform of D_n obeys

$$\lim_{\beta\to\infty}\frac{1}{\beta^2}\ln\int \mathrm{d}\varepsilon\, D_n(\varepsilon+\varepsilon_n)\,\mathrm{e}^{-\beta\varepsilon}=\tfrac{1}{2}\Gamma_n^2. \tag{18}$$

Since the optimized lower bound in (8) for $f(\varepsilon)=\exp\{-\beta\varepsilon\}$ takes on the desired limit, there remains the construction of a suitable upper bound. The quasi-classical upper bound in (8) does not suffice. The construction proceeds in several steps. The Laplace transform of the density of states corresponding to the restricted cut-off potential $E_nV_rE_n$ is bounded from above by the Rayleigh–Ritz principle applied to $E_nV_rE_n$. The average of this bound with respect to the random potential V is represented as a functional integral over V. The substitution $V=:\beta v$ transforms the integral into a form which allows for its asymptotic evaluation for $\beta\to\infty$ by Laplace's method in function space [28]. The resulting upper limit analogous to (18) tends to $\Gamma_n^2/2$ as $r\to\infty$.

Remark. The result (17), especially when combined with (15) and (7) in the case (6), considerably generalizes earlier results [29, 30]. In [29] the decay constant Γ_n for Gaussian white noise has correctly been found by optimizing $\gamma_{n,\psi}^2$ over a restricted set of $|\psi\rangle$-vectors. In [30], by varying over this set for $n=0$, the decay constant $\Gamma_0(\lambda^2)$ has correctly been found for the Gaussian covariance (6), but it has erroneously been stated there that $\Gamma_0^2=\gamma_{0,\psi_0}^2$ for all isotropic covariances.

Having discussed the restricted density of states D_n to some extent, we return to general homogeneous random potentials with $\bar{V}=0$ and address ourselves to the question, in which sense $\Sigma_{n=0}^{\infty} D_n$ may serve to approximate the unrestricted density of states D. To this end, following [11, 13], we introduce the densities $\hat{D}_n(\varepsilon):=\langle x|\overline{E_n\delta(\varepsilon-H)E_n}|x\rangle$.

We note that $2\pi l^2 \hat{D}_n$ is a probability density on the real line with first moment ε_n and variance $C(0)$. Unlike D_n the density $\hat{D}_n$ is in general not symmetric with respect to ε_n, if V and $-V$ have the same distribution. For Gaussian random potentials with $C(0)<\infty$ we suspect that $\hat{D}_n$ like D has a Gaussian tail for $\varepsilon \to -\infty$ with decay constant $[C(0)]^{1/2}$.

Concerning the relation between D_n and $\hat{D}_n$ for general random potentials we state the inequality

$$\int \mathrm{d}\varepsilon\, D_n(\varepsilon) f(\varepsilon) \leqslant \int \mathrm{d}\varepsilon\, \hat{D}_n(\varepsilon) f(\varepsilon) \tag{19}$$

formally following for $f(\varepsilon_n)=0$ from (12) with $A = H_0 + V_r$ and $E = E_n$ after averaging, dividing by πr^2, and taking the limit $r \to \infty$. For the actual proof one has to show that

$$\lim_{r\to\infty} \frac{1}{\pi r^2} \int \mathrm{d}^2 x\ \Theta(|x| - r) \langle x | E_n \overline{f(H_0 + V_r)} E_n | x \rangle = 0. \tag{20}$$

We conjecture that (20) generally holds provided that $\overline{f(V(0))} < \infty$. So far we only have a proof for the case $f(\varepsilon) = \exp\{-\beta\varepsilon\} - \exp\{-\beta\varepsilon_n\}$ which is based on an upper bound derivable from the Feynman–Kac–Itô formula [28].

In the following remarks we assume that the random potential is not only homogeneous but also isotropic.

Remarks

Since the unrestricted density of states can be reconstructed [11, 13] according to $D = \Sigma_{n=0}^{\infty} \hat{D}_n$, the inequality (19) yields lower bounds to $\int \mathrm{d}\varepsilon\, D(\varepsilon) f(\varepsilon)$ for convex functions f. In particular, the grand-canonical potential corresponding to the unrestricted one-electron Hamiltonian H is overestimated when neglecting level mixing.

Defining the absence of level mixing as the validity of $\Sigma_{n=0}^{\infty} D_n = D$, the inequality (19) with $f(\varepsilon) = \exp\{-\beta\varepsilon\}$ shows by the invertibility of the Laplace transformation that level mixing does not occur if and only if $\hat{D}_n = D_n$ for all n. The preceding discussion of the widths and tails of D_n and $\hat{D}_n$ suggests that the approximation $\hat{D}_n \approx D_n$ can only be reasonable, if $(2n+1) l^2 \ll \lambda^2$ is fulfilled besides the usual condition $C(0) \ll (2\varepsilon_0)^2$. Here λ is the (smallest) correlation length of the covariance C.

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